MATH 20D Spring 2023 Lecture 9.

Conjugate roots, Free Mechanical Vibrations

Outline

1. More on the case of complex roots.

2. Free Mechanical Vibrations

Contents

Last Time

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Solve the initial value problem

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- So $y(t) = \frac{1}{\sqrt{3}}e^{-t}\sin(\sqrt{3}t)$ solves the IVP.

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Suppose $b^2 - 4ac < 0$ and consider the general solution to the ODE (2) given by

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Example

(a) Solve the IVP $\frac{1}{8}y''(t) + 16y(t) = 0$, $y(0) = 1/2$, $y'(0) = -\sqrt{2}$.
(b) Rewrite your solution to (a) in the form $y(t) = Ae^{\alpha t}\sin(\beta t + \phi)$.

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